Imperial College London

Lecture 13

Impulse Response & Filters

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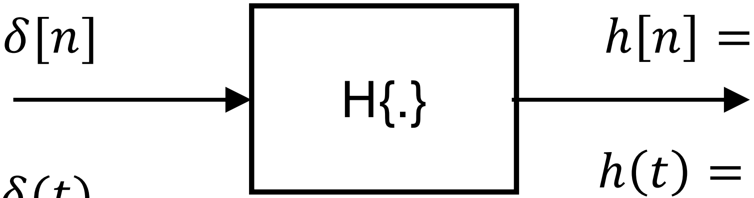
Impulse Response of a Discrete System

- Apply a discrete impulse *x[n]* to the input of a discrete time system, the output *y[n]* is known as the system's **impulse response *h[n]*.**

$x[n] = \delta[n]$

$x(t) = \delta(t)$



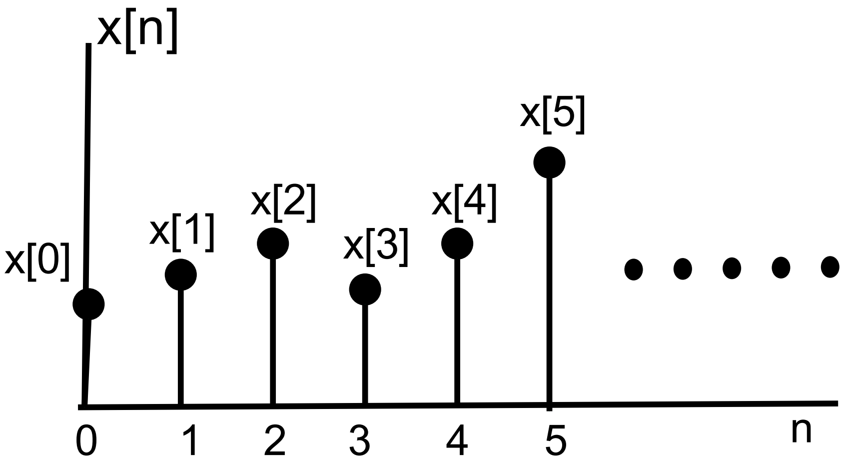
$h[n] = \text{discrete impulse response}$

$h(t) = \text{continuous impulse response}$

- The impulse response of a **linear** system defines and characterises the system completely – both its transient behaviour and its frequency response.
- This applies to both continuous time and discrete time linear systems.
- An impulse signal contains ALL frequency components (L3, S7). Therefore, applying an impulse to input stimulates the systems at ALL frequencies.
- Since integrating a unit impulse = a unit step function, we can obtain the step response of the system by integrating the impulse response.

Discrete signal as sum of weighted impulses

- Remember from L10, S7, we can represent a causal discrete signal *x[n]* in terms of sum of weighted delayed impulses:



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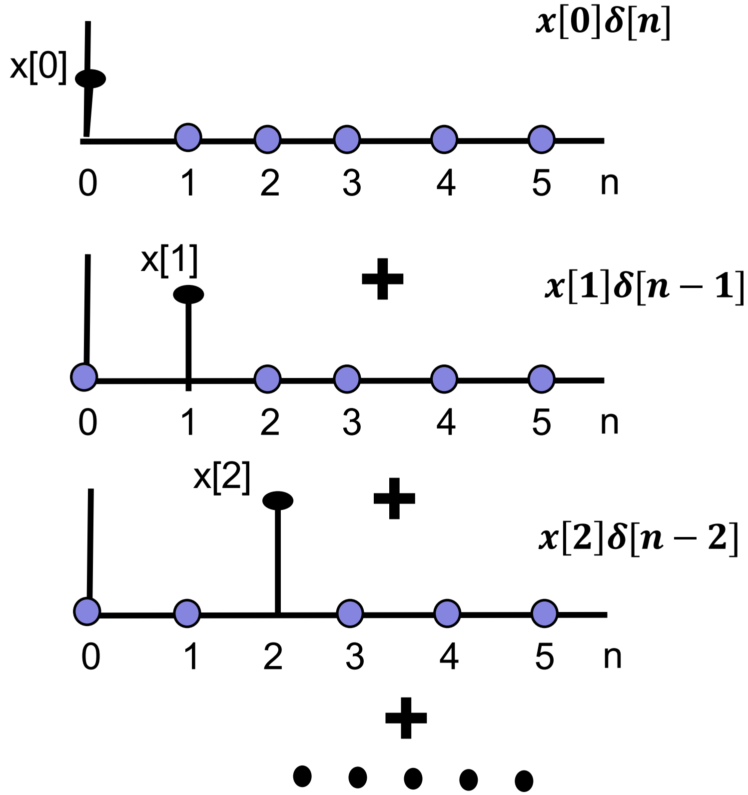
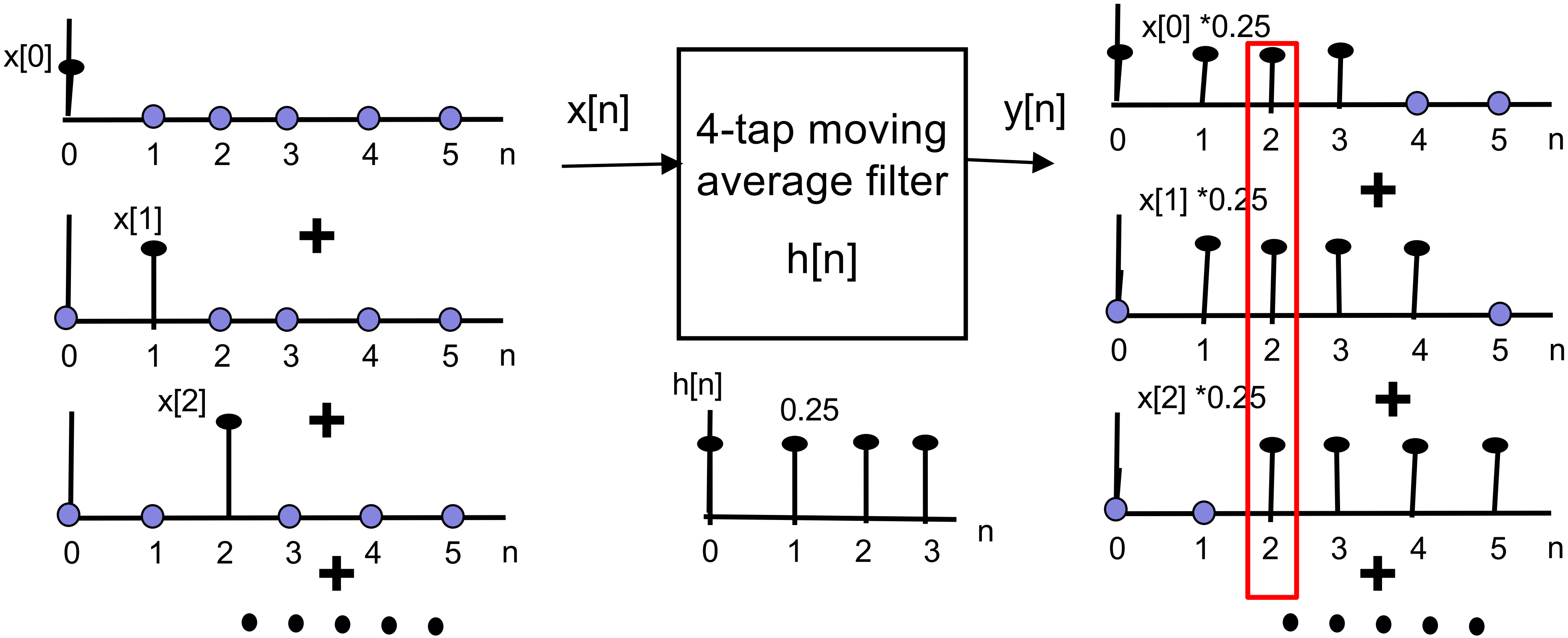


$$x[n] = \sum_{k=0}^{\infty} x[k]\,\delta[n-k]$$

Impulse Response and Convolution

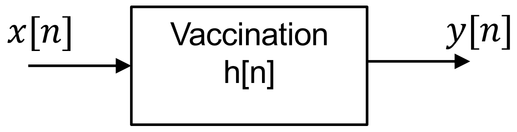


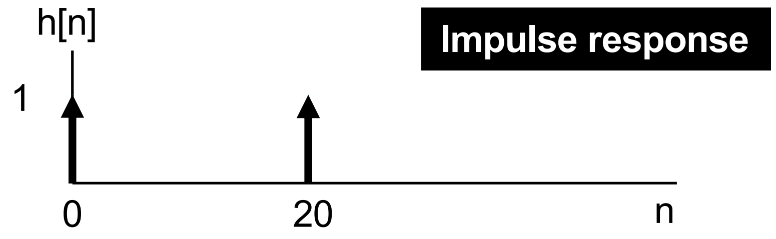
- We can therefore derive the output of a discrete linear system, by adding together the system's response to EACH input sample separately.
- This operation is known as **convolution**:

$$y[n] = x[n] * h[n] = \sum_{m=0}^{\infty} x[m]h[n-m]$$

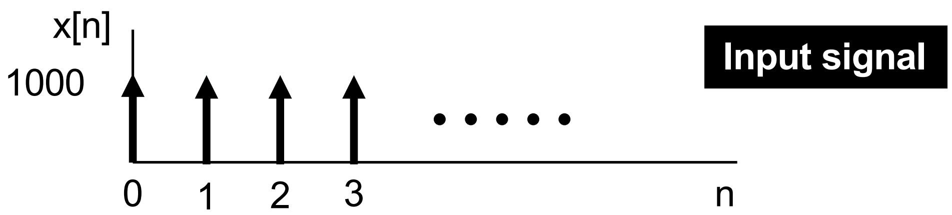
Convolution example - COVID vaccination

- Covid vaccine require two doses, 3 weeks apart. Impulse response h[n] for this vaccination system is:



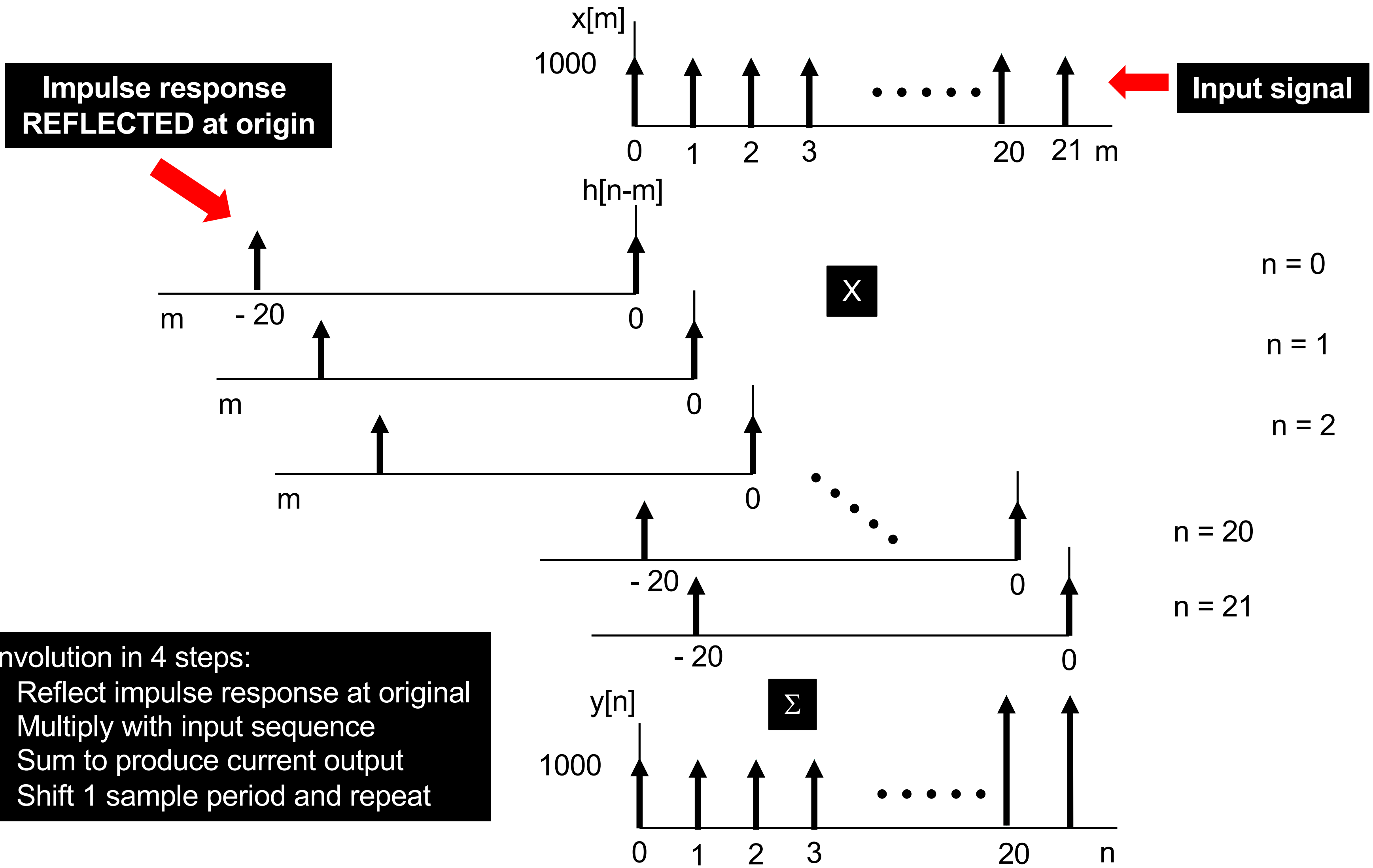


- UK started vaccination of its elderly population with a plan of vaccinating, say, 1000 people per day after the first day (day 0). The input x[n] is:



- How many doses would NHS need to provide from day 0? (i.e. y[n])

Graphical representation of Convolution

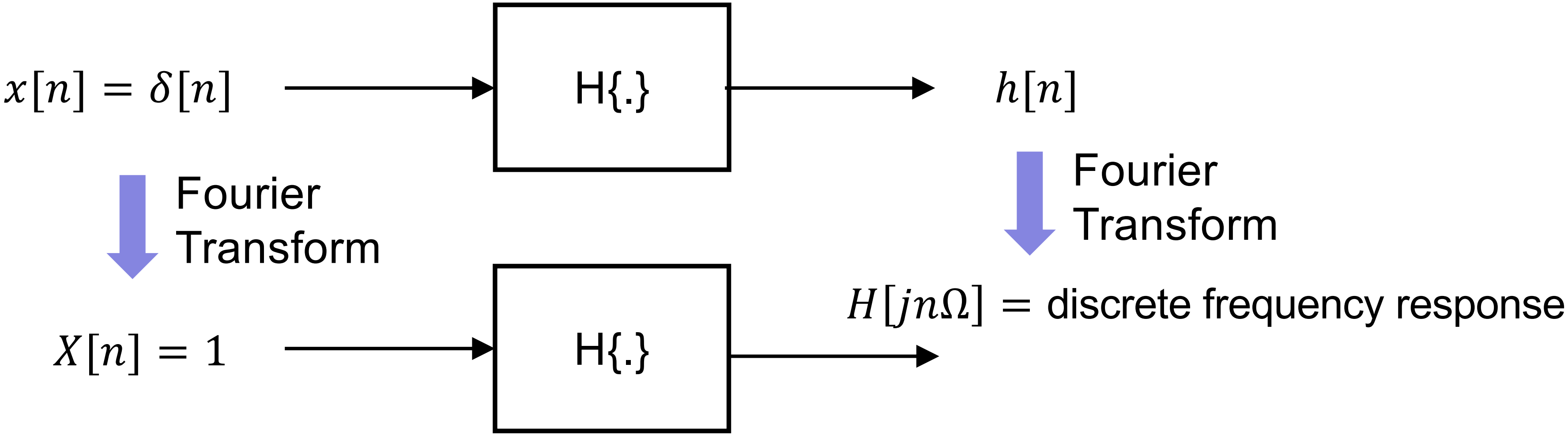


Convolution in 4 steps:

1. Reflect impulse response at original
2. Multiply with input sequence
3. Sum to produce current output
4. Shift 1 sample period and repeat

Impulse Response & Frequency Response

- Since a unit impulse contains all frequency, and its Fourier transform is a constant at 1 (see L3, S7), the Fourier transform of the impulse response h[n] or h(t) give us the systems' frequency response:



- This applies to both continuous time and discrete time linear systems.
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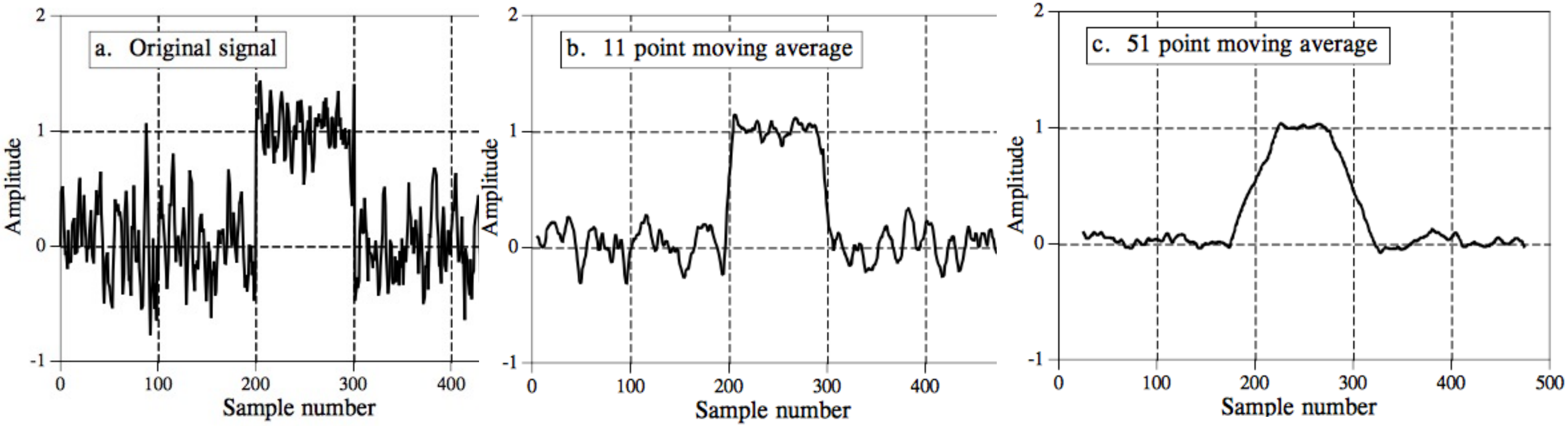
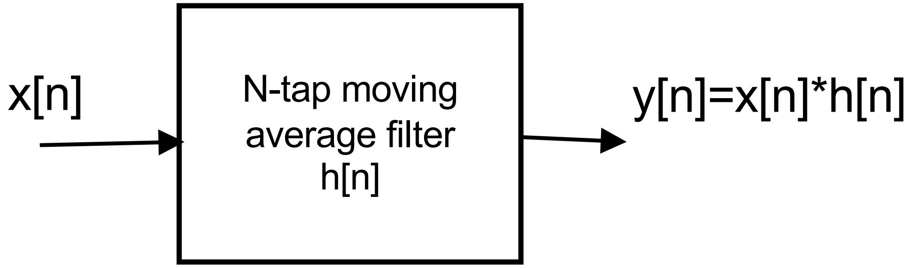
Moving Average Filter = FIR lowpass filter



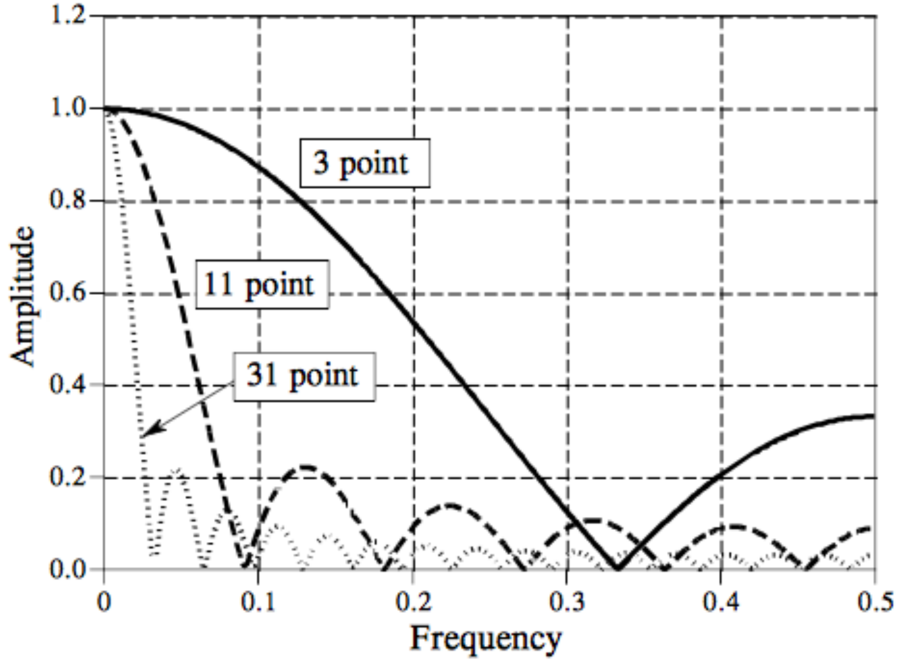


- N-tap (or N point) moving average filter – higher N, lower the cut off frequency
- For a N-tap moving average filter, it impulse response has N impulses.
- If input x[n] has M non-zero samples (i.e. finite length), output y[n] is also finite in length, and has M+N non-zero samples. Hence the name Finite Impulse Response (FIR) filter.

Frequency Response of N-tap moving average filter

- The impulse response of a moving average filter is a rectangular pulse.
- The Fourier transform of a rectangular pulse is of the form (sin x)/x or sinc(x) function (see Lecture 3 slide 6) in the case of continuous time.
- For discrete time case, the frequency response of a moving average filter with N taps (or points) is:

$$H[f] = \frac{\sin(fN)}{N \sin f}$$



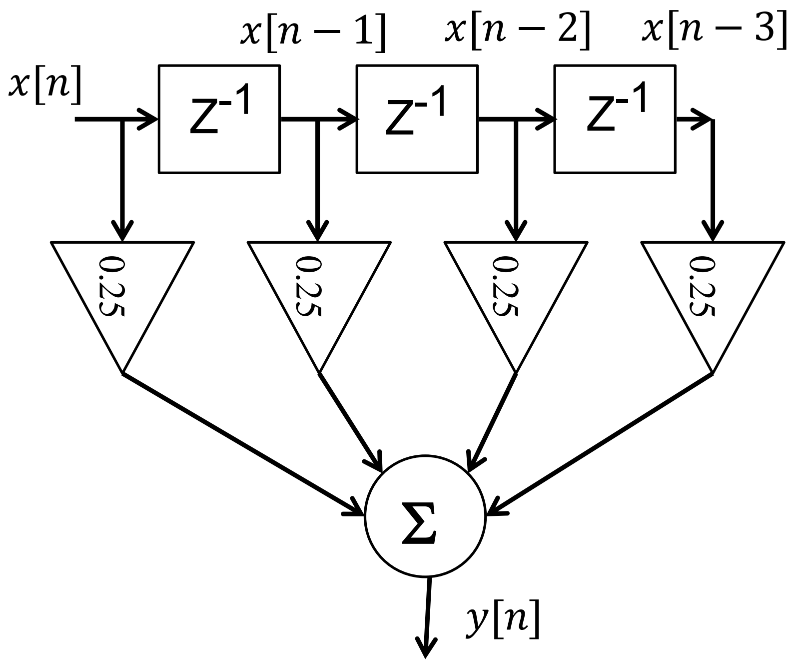
- Here f is normalised to 0 -> 0.5 x sampling frequency *fs*.

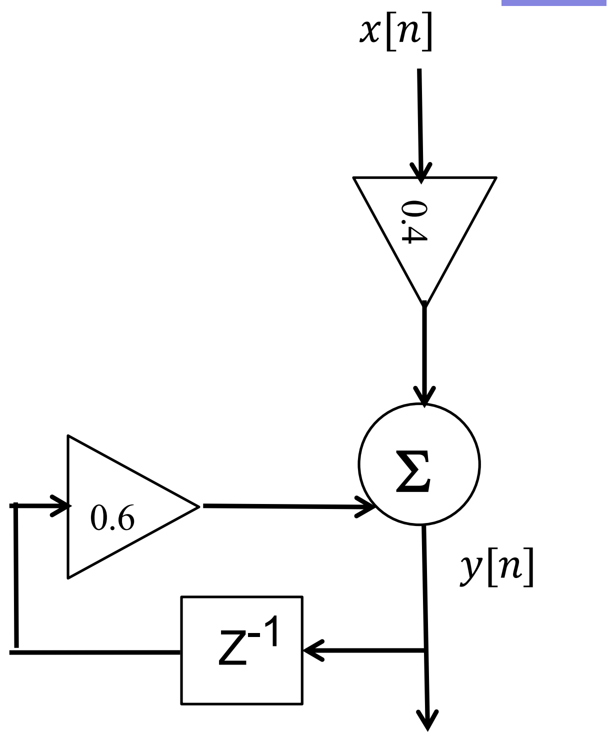
Recursive or Infinite Impulse Response Filter

FIR

x[n] =
{1.0, 1.0, 1.0, 1.0, 1.1, 0.8, 1.2, 0.9, 1.0, 1.2, 0.9, …}

IIR

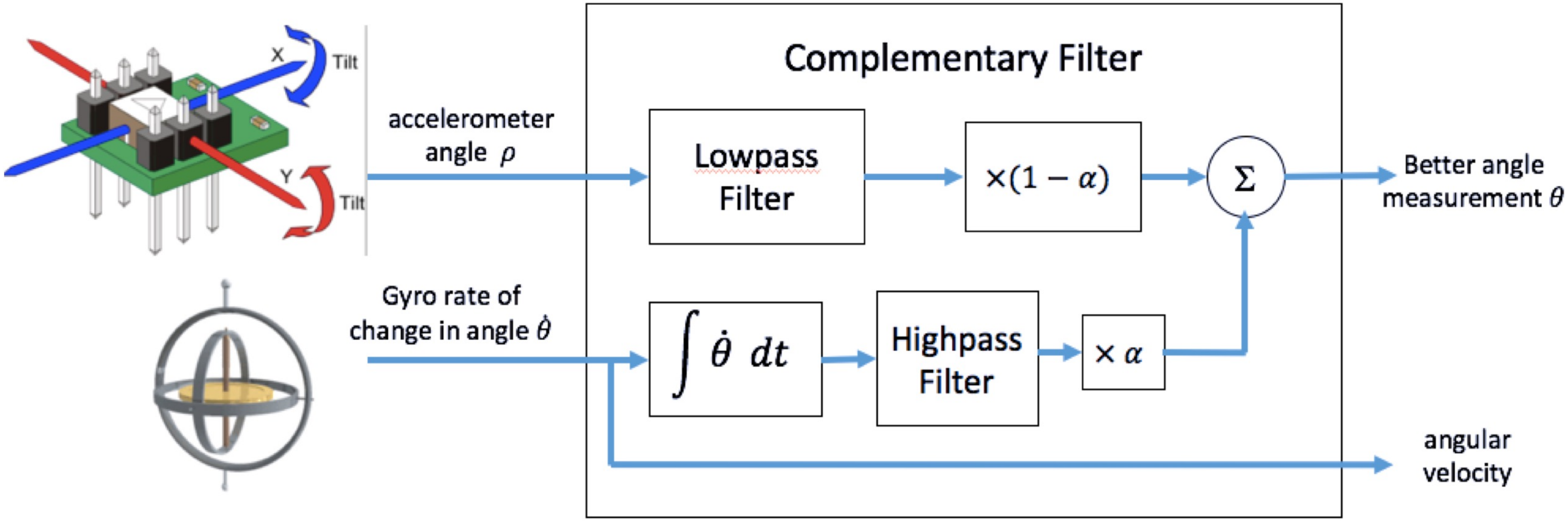




y[n] = (0.25, 0.5, 0.75, 1.0, 1.025,
0.975, 1.025, 1.0, 1.0, 1.075, 1.0, …..}

y[n] = (0.4, 0.64, 0.784, 0.87, 0.962,
0.897, 1.018, 0.971, 0.983, 1.07,
1.002…..}

Complementary Filter used with IMU

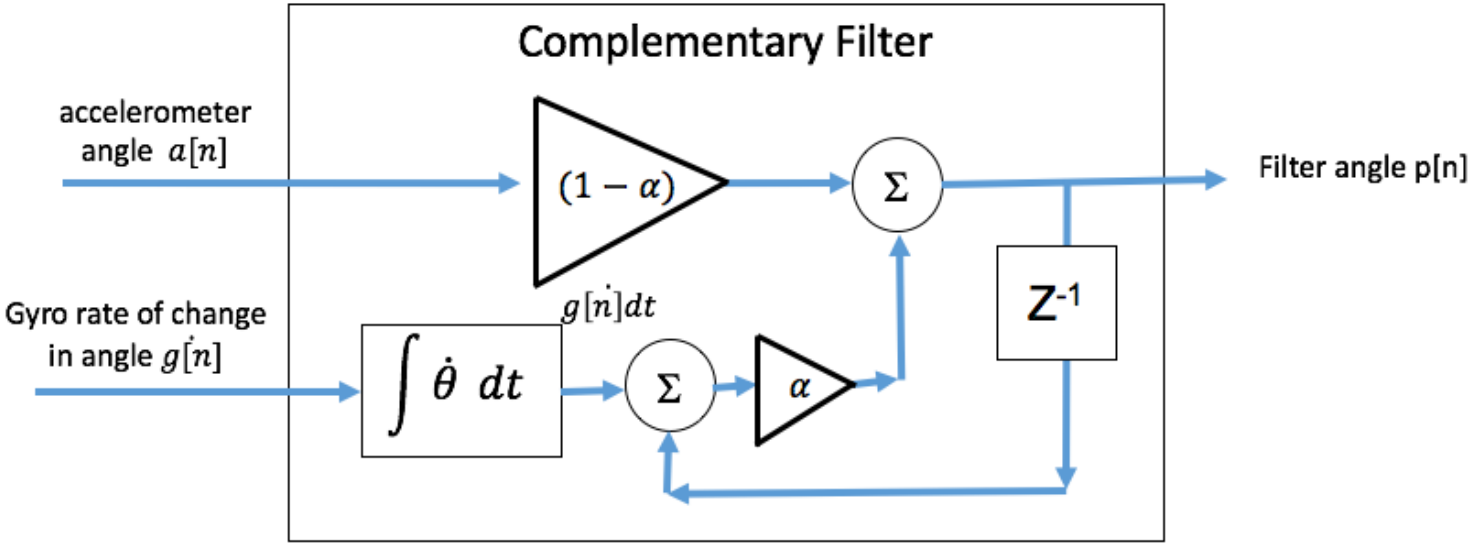


$$\text{angle } \theta_{new} = \alpha \times (\ \theta_{old} + \ \dot{\theta}\ dt) + (1-\alpha) \times \rho$$

where

α = scaling factor chosen by users and is typically between 0.7 and 0.98
ρ = accelerometer angle
θ_{new} = new output angle
θ_{old} = previous output angle
$\dot{\theta}$ = gyroscope reading of the rate of change in angle
dt = time interval between gyro readings

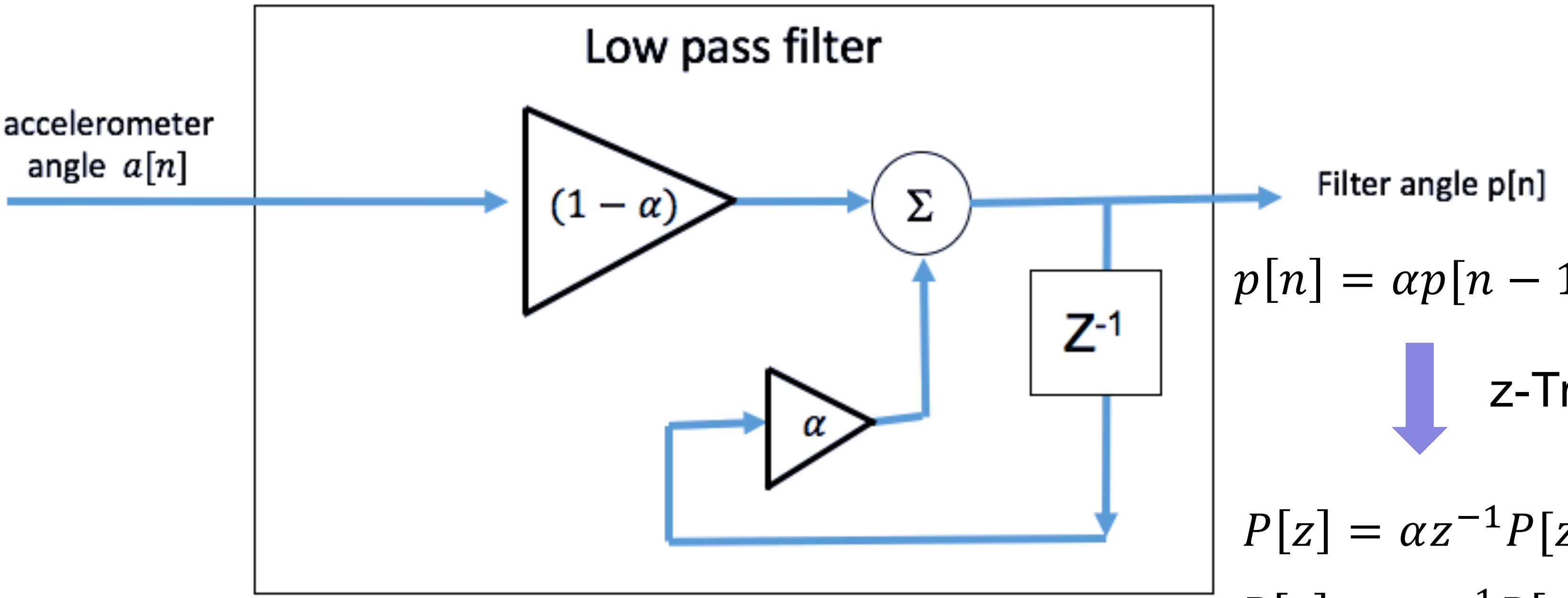
Signal flow diagram model



$$p[n] = \alpha(p[n-1] + g[n]\times dt) + (1-\alpha)a[n]$$

```
def read_imu(dt):
    global g_pitch
    alpha = 0.7     # larger = longer time constant
    pitch = int(imu.pitch())
    roll = int(imu.roll())
    g_pitch = alpha*(g_pitch + imu.get_gy()*dt*0.001) + (1-alpha)*pitch
```

Lowpass filter the accelerometer data



$$p[n] = \alpha p[n-1] + (1-\alpha)a[n]$$

z-Transform

$$P[z] = \alpha z^{-1}P[z] + (1-\alpha)A[z]$$

$$P[z] - \alpha z^{-1}P[z] = (1-\alpha)A[z]$$

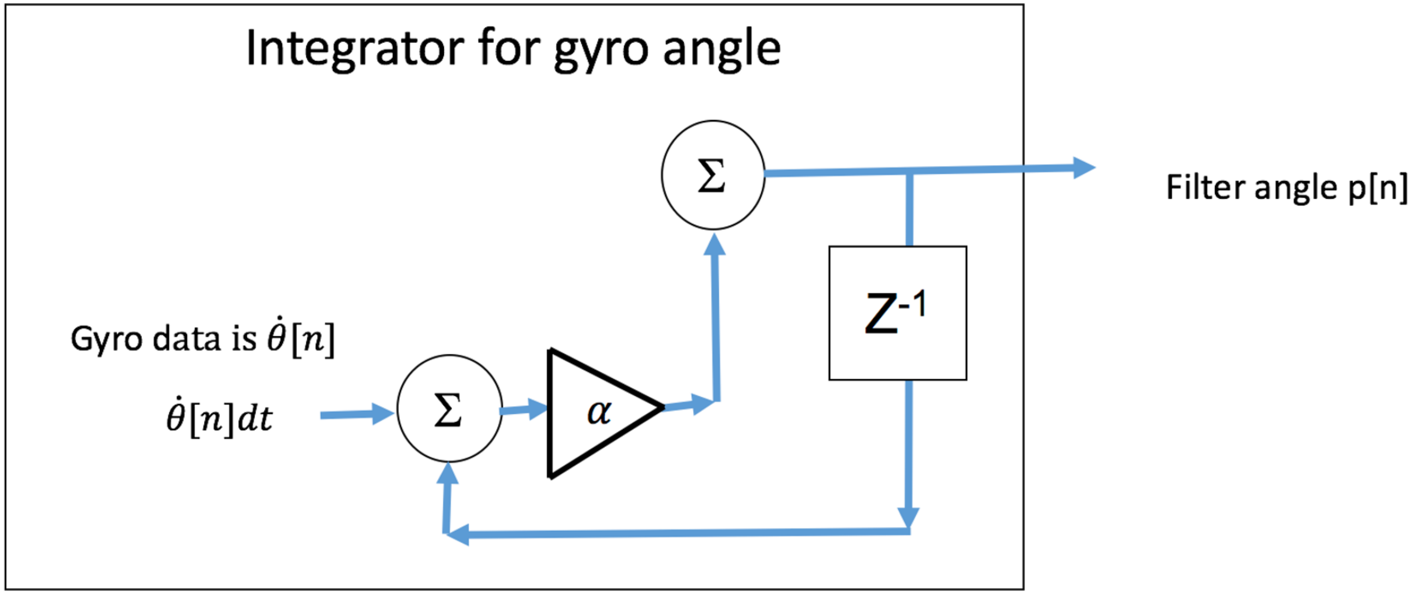
$$H[z] = P[z]/A[z] = \frac{1-\alpha}{1-\alpha z^{-1}}$$

- Now assume that gyroscope reading is zero (i.e. steady state tilt), g[n] = 0.
- Now the system is exactly the same as that in Lecture 12, slide 12.
- Therefore the accelerometer data a[n] is lowpass filtered!

Time Constant $\tau \approx \frac{\alpha}{1-\alpha} dt$

and dt is the sampling period

Integrating the gyroscope reading



$$p[n] = \alpha(p[n-1] + \dot{\theta}[n]dt)$$

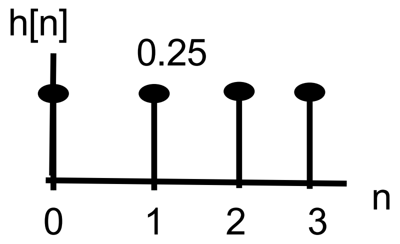
$$p[n] = \alpha^n p[0] + \ \alpha^n k$$

- Assume the gyro is not moving, but has a constant offset $\dot{\theta}_e$.
- dt is also constant.
- If $\alpha = 1$, $p[n]$ is a ramp with a gradient of $\dot{\theta}_e$.
- If $\alpha < 1$, then the effect of the error overtime diminishes to $\alpha^n \rightarrow 0$.

Three Big Ideas (1)

1. A discrete time system can be characterized by its impulse response:

$$h[n] = b_0\delta[n] + b_1\delta[n-1] + b_2\delta[n-2] \ldots. + b_k\delta[n-k]$$



Impulse response of a 4-tap moving average filter

2. Once we know the impulse response $h[n]$, and the input sequence $x[n]$, we can find the output $y[n]$ by convolution:

$$y[n] = x[n] * h[n] = \sum_{m=0}^{\infty} x[m]h[n-m]$$

Three Big Ideas (2)

3. Convolution operation can be performed in four steps:

   To obtain output y[n]:

   1) **Reflect** impulse response at n to get h[n-m]
   2) **Multiply** input sequence x[m] with h[n-m]
   3) **Sum** the product of the two sequences to get one output y[n]

$$y[n] = \sum_{m=0}^{\infty} x[m]h[n-m]$$

   4) **Advance** the reflected impulse response by one sample period and repeat to get the **next** y[n]